Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00016-2
J. Appl. Maths Mechs, Vol. 63, No. 6, pp. 985-987, 1999

0021-8928/99/$—see front matter

THE STABILITY OF THE JACOBI ELLIPSOIDS OF A ROTATING FLUID†

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(*Received* 22 *February* 1999)

Using the method developed in [1] for solving the problem of the conditional stability of MacLaurin ellipsoids, it is proved that Jacobi ellipsoids are stable under Dirichlet conditions for all case when Jacobian ellipsoids exist.

It is well known that, in general, the problem of the stability of ellipsoidal, equilibrium figures has been formulated and solved by Lyapunov for the MacLaurin and Jacobi cases.

Lyapunov's proofs are based on the application of his own type of analogue of Routh's theorem and reduce to proofs of the existence of a strict minimum of the functional of the transformed potential energy for the steady rotation which is investigated in each case. By applying this method, Lyapunov, in particular, shows that Jacobi ellipsoids are always stable subject to the condition that the perturbation satisfies the Dirichlet assumption on the ellipsoidal form of the fluid surface and the homogeneous vortex nature of the perturbed velocity field $^1/_2 \operatorname{rot} \upsilon = \{\omega_1(t), \omega_2(t), \omega_3(t)\}$.

However, it was shown in [1] that, in the case of perturbations which satisfy the Dirichlet assumptions, the initial dynamical system in functional space becomes a system of ordinary differential equations for the components of rot υ, the semi-axes of the ellipsoid a, b and c and the components of the angular velocity p, q and r in a moving frame of reference as functions of time. This, in turn, enables one to use the second Lyapunov method to solve the problem of the stability of the corresponding steady solutions, which avoids the need to carry out complex calculations associated with the study of the functional of the transformed potential energy for an extremum.

The system of ordinary differential equations, into which the initial system in ordinary and partial derivatives is converted and which describes the dynamics of a self-gravitating fluid, has four first integrals (of the energy, the momenta, the constancy of the vortex intensity and mass)

$$\frac{1}{2}(A_1p^2+B_1q^2+C_1r^2)+\frac{1}{2}(A_2\omega_1^2+B_2\omega_2^2+C_2\omega_3^2)+W+\frac{M}{10}(\dot a^2+\dot b^2+\dot c^2)=\text{const} \quad (1)$$

$$(A_1p+A_2\omega_1)^2+(B_1q+B_2\omega_2)^2+(C_1r+C_2\omega_3)^2=\text{const} \quad (2)$$

$$(\omega_1/a)^2+(\omega_2/b)^2+(\omega_3/c)^2=\text{const} \quad (3)$$

$$abc=a_0b_0c_0 \quad (4)$$

Here

$$W=-\frac{2}{5}MH,\quad H=\frac{3}{4}M\int\frac{d\lambda}{\sqrt{\varphi(\lambda)}},\quad \varphi(\lambda)=(a^2+\lambda)(b^2+\lambda)(c^2+\lambda) \quad (5)$$

$$A_1=\frac{M}{5}\frac{(b^2-c^2)^2}{b^2+c^2},\quad A_2=\frac{4M}{5}\frac{b^2c^2}{b^2+c^2}\ (ABC,\ abc)$$

where M is the mass of the fluid.

This system admits of the particular solution

$$x_0=\{p=q=\omega_1=\omega_2=0\quad r=\omega_3=\Omega;\quad a=a_0,\ b=b_0,\ c=c_0\} \quad (6)$$

which describes a Jacobi ellipsoid. A proof that this is stable (under the conditions mentioned above) will be given below.

We know that the three parameters: Ω, b_0/a_0, c_0/a_0 of a Jacobi ellipsoid are related by the two equations

†*Prikl. Mat. Mekh.* Vol. 63, No. 6, pp. 1052–1054, 1999.

$$\Omega^2 = P - R\frac{c_0^2}{a_0^2} = Q - R\frac{c_0^2}{b_0^2} \tag{7}$$

where

$$P = -\frac{2}{a_0}\frac{\partial H}{\partial a}, \quad R = -\frac{2}{c_0}\frac{\partial H}{\partial c}, \quad Q = -\frac{2}{b_0}\frac{\partial H}{\partial b} \tag{8}$$

We take solution (6) for the unperturbed motion. In the perturbed motion, we put

$$a = a_0 + \alpha, \quad b = b_0 + \beta, \quad r = \Omega + \xi, \quad \omega_3 = \Omega + \eta \tag{9}$$

and retain the previous notation for the remaining variables. Substituting (9), taking account of relation (4), into integrals (1)–(3), we obtain the first integrals, corresponding to them, of the equations of the unperturbed motion, denoting them by V_1, V_2 and V_3, respectively.

We now consider the function

$$V = 2S_0\tilde{V}_1 - \tilde{V}_2 + n\tilde{V}_3^2 + \nu\tilde{V}_2^2 \tag{10}$$

where

$$S = C_1 + C_2 = \frac{M}{5}(a^2 + b^2), \quad \tilde{V}_i = V_i(X) - V_i(X_0)$$

where n and ν are certain real positive numbers.

Taking account of the form of integrals (1)–(3), in the case of V (10) in a small neighbourhood X_0, we have, apart from second-order terms inclusive

$$V = \left[2S_0\left(d\tilde{W} + d\frac{C_1r^2 + C_2\omega_3^2}{2}\right) - 2S_0\Omega_0 d(C_1r + C_2\omega_3)\right] + K_1 + K_2 \tag{11}$$

$$K_1 = 2S_0\frac{M}{10}(\dot{a}^2 + \dot{b}^2 + \dot{c}^2) + 2S_0\left(\frac{A_{10}p^2}{2} + \frac{A_{20}\omega_1^2}{2}\right) - (A_{10}p + A_{20}\omega_1)^2 +$$

$$+ S_0B_{10}q^2 + S_0B_{20}\omega_2^2 - (B_{10}q + B_{20}\omega_2)^2$$

$$K_2 = 2S_0\left(d^2\tilde{W} + \frac{\Omega^2}{2}d^2S + \frac{C_{10}}{2}\xi^2 + \frac{C_{20}}{2}\eta^2\right) + n(dV_3)^2 + \left(\nu - \frac{1}{2S_0^2\Omega^2}\right)(dV_2)^2$$

Here, $\tilde{W} = W|_{c=a_0b_0c_0/(ab)}$, dV_2 and dV_3 are the linear parts of the functions V_2 and V_3 which correspond to the increments (9) of the independent variables

$$dV_2 = z = 2S_0\Omega d(C_1r + C_2\omega_3)$$

$$dV_3 = y = 2\frac{\Omega}{C_0}d\frac{\omega_3}{C} = 2\frac{\Omega}{C_0^2}\left(\eta + \Omega\frac{\alpha}{a_0} + \Omega\frac{\beta}{b_0}\right) \tag{12}$$

The linear part of V (10), written in the square brackets, is zero by virtue of Eqs (7). The quadratic form with respect to $\dot{a}$, $\dot{b}$, $\dot{c}$, p, q, ω_1, ω_2, K_1 is positive definite by virtue of the fact that $c_0 < a_0$, $c_0 < b_0$ and by the definitions of A_i and B_i and S. The form of K_2 is the form with respect to the remaining variables α, β, ξ, η. We shall consider it here but, for convenience, as the form with respect to 1053n, where z and y are defined by formulae (2), rather than the form with respect to (α, β, ξ, η). This has no effect whatsoever on the result since the property of positive definiteness is invariant with respect to a no-degenerate change of variables. If account is taken of expression (12), the form of K_2 in the variables (α, β, z, y), takes the form

$$S_0\left(\left(\tilde{W}_{aa}|_{\tilde{X}_0} + 3\frac{M}{5}\Omega^2\right)\alpha^2 + \left(\tilde{W}_{bb}|_{\tilde{X}_0} + 3\frac{M}{5}\Omega^2\right)\beta^2 + 2\tilde{W}_{ab}|_{\tilde{X}_0}\alpha\beta\right) + ny^2 + \left(\nu - \frac{1}{2S_0^2\Omega^2}\right)z^2 + \ldots \tag{13}$$

Here, $\tilde{X}_0 = (a_0, b_0)$, and the products of αz, αy, βz, βy by the bounded coefficients which depend on a_0 and b_0 are denoted by dots. It is clear that, in the case of a positive-definite form with respect to α and β with a matrix

$$\left\| \begin{matrix} \tilde{W}_{aa}|_{\tilde{X}_0} + \frac{3M}{5}\Omega^2 & \tilde{W}_{ab}|_{\tilde{X}_0} \\ \tilde{W}_{ab}|_{\tilde{X}_0} & \tilde{W}_{bb}|_{\tilde{X}_0} + \frac{3M}{5}\Omega^2 \end{matrix} \right\| \tag{14}$$

the numbers n and ν can be chosen such that the form of (13) turns out to be positive definite as well as V (10) with these values of (n, ν). This will mean that V (10) is the Lyapunov function for this problem and that the steady rotation (6) is stable.

It is easily shown using elementary calculations that matrix (14) is positive definite.

Taking expression (5) for $W(a, b, c)$ and the expression for $\tilde{W}(a, b)$ into account, we obtain, after differentiation, in explicit form: the second derivatives of W at the point (a_0, b_0) appearing in (14) and, also, starting from Eqs (7), an expression for Ω^2.

Using these expressions, we obtain formulae for the elements m_{ij} $(i, j = 1, 2)$, $m_{22} = m_{11}|_{a_0=b_0}$, $m_{12} = m_{21}$ of matrix (14). In this case, as is easily verified, it is found that

$$m_{11} \pm \frac{b_0}{a_0} m_{12} = \frac{2\cdot 3}{5\cdot 4} M^2 \int \frac{D_1^{\pm}(\lambda)}{\varphi^{3/2} a_0^2 (c_0^2 + \lambda)(a_0^2 + \lambda)} d\lambda$$

$$m_{22} \pm \frac{b_0}{a_0} m_{12} = \frac{2\cdot 3}{5\cdot 4} M^2 \int \frac{D_2^{\pm}(\lambda)}{\varphi^{3/2} b_0^2 (c_0^2 + \lambda)(b_0^2 + \lambda)} d\lambda$$

where $D_1^{\pm}$ are two polynomials of the fourth degree in λ with coefficients which are functions of a_0, b_0, c_0, and all of these coefficients are positive when $c_0 < a_0$, $c_0 < b_0$; the same also applies to $D_2^{\pm} = D_1^{\pm}|_{a_0=b_0}$.

The relations

$$\tilde{W}_{aa}^0 + \frac{3M}{5}\Omega^2 > \tilde{W}_{ab}^0 \frac{b_0}{a_0}, \quad \tilde{W}_{bb}^0 + \frac{3M}{5}\Omega^2 > \tilde{W}_{ab}^0 \frac{a_0}{b_0}$$

$$\tilde{W}_{aa}^0 + \frac{3M}{5}\Omega^2 > -\tilde{W}_{ab}^0 \frac{b_0}{a_0}, \quad \tilde{W}_{bb}^0 + \frac{3M}{5}\Omega^2 > -\tilde{W}_{ab}^0 \frac{a_0}{b_0}$$

therefore hold. It follows from these that, first, both diagonal elements of (14) are positive and, second, that

$$\left(\tilde{W}_{aa}^0 + \frac{3M}{5}\Omega^2\right)\left(\tilde{W}_{bb}^0 + \frac{3M}{5}\Omega^2\right) > \left(\tilde{W}_{ab}^0\right)^2$$

So that (14) is a positive-definite matrix. In turn, the positive definiteness of the form (13) follows from this, and this means that V (10), for certain sufficiently large n and ν, is greater than zero. Hence, (6) is a stable solution of the system of ordinary differential equations which the initial system with an infinite number of degrees of freedom in the case of Dirichlet conditions becomes, and the Jacobi ellipsoids are always conditionally stable (with respect to $a, b, c, p, q, r, \omega_1, \omega_2, \omega_3$) when they are defined (that is, when the eccentricities $l = \sqrt{(a^2 - c^2)}/a \geqslant l_0 = 0.8126$).

REFERENCES

1. RUMYANTSEV, V. V., On the stability of the MacLaurin ellipsoids of a rotating fluid. *Prikl. Mat. Mekh.*, 1959, **23**, 3, 494–504.
2. APPELL, P., *Traité mécanique rationelle. T. 4. Fasc. 1., Figures d'équilibre d'une masse liquide homogene en rotation*. Gauthier-Villars, Paris, 1932.

Translated by E.L.S.